Comparison of Partially Penetrated Open- and Closed-Ended Vertical Drains

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ABSTRACT: The bottom surface of the ordinary sand drains is pervious. However, the bottom of the plastic vertical drains can be considered as an impervious end. Two kinds of analytical solutions for consolidation by partially penetrated open- and closed-ended vertical drains are presented in this paper. The basic equations, the continuity conditions, the orthogonality, the solutions process and the final results and under two kinds of drainage conditions of the end of the vertical drains are compared all-sidedly.

Keywords: consolidation; vertical drains; sand drains; soft marine clay; marine sediment

1 INTRODUCTION

In recent years, the sites with good foundation have already been used up and civil engineers have no alternative but to utilize soft ground areas with deep marine deposits such as reclaimed land, coastal lowlands, swampy sites, etc. When the soft clay layer is very deep or surcharge loading is too small, it is not economical to penetrate the vertical drains to full depth. The consolidation problem of ground with partially penetrated vertical drains is analyzed mainly by numerical methods [1,2]. The author [3,4] has obtained an analytical solution for the above problem considering the ordinary sand drains only. Normally, because the cross section of the ordinary sand drains (SD) is big, the end of SD can be regarded as a pervious end or an open end. However, because the cross section of the plastic board vertical drains (PVD) is very small and sometimes the end is filled with soil, the end of PVD can be regarded as an impervious end or a closed end. Therefore, it is necessary to study the basic equations, the continuity conditions, and the solutions for the partially penetrated vertical drains with the closed end and to compare those with the open end of the vertical drains.

2 MATHEMATICAL MODELING

For easier discussion, similar to Hart et al. [5] and Runesson et al. [2], the horizontal plane through the bottom of the vertical drains is called the plane of penetration. The sections above and below the plane of penetration are defined as the improved section and the unimproved section, respectively. Although the continuity conditions near the plane of penetration are very complex, from the view of total balance of the quantity of water flow and the pore pressure, the following assumptions are made:

For the open-ended vertical drains:
  i) The pore pressure within the vertical drains is equal to the pore pressure of the unimproved section.
  ii) The average pore pressure of the soil of the improved section is equal to the pore pressure of the unimproved section.
  iii) The total water flow of the vertical drains and the soil zone of the improved section are equal to the water flow of the unimproved section.
For the closed-ended vertical drains:
  iv) The vertical direction of water flow within the vertical drains is equal to zero.
  v) The average pore pressure of the soil of the improved section is equal to the pore pressure of the unimproved section.
  vi) The water flow of the soil zone of the improved section (except the area of the vertical drains) is equal to the water flow of the unimproved section.
The basic equations are the same, they are listed as Eqs. (1) ~ (5)

For the improved section \((0 \leq z \leq h_i)\),

Smear zone, \(r_w \leq r \leq r_s\):

\[
\frac{k_i}{m_i \gamma_w} \left( \frac{1}{r} \frac{\partial u_{i1}}{\partial r} + \frac{\partial^2 u_{i1}}{\partial r^2} \right) + \frac{k_i}{m_i \gamma_w} \frac{\partial^2 \bar{u}_i}{\partial z^2} = \frac{\partial \bar{u}_i}{\partial t} \tag{1}
\]

Natural soil, \(r_s \leq r \leq r_e\):

\[
\frac{k_h}{m_i \gamma_w} \left( \frac{1}{r} \frac{\partial u_{h1}}{\partial r} + \frac{\partial^2 u_{h1}}{\partial r^2} \right) + \frac{k_v}{m_i \gamma_w} \frac{\partial^2 \bar{u}_i}{\partial z^2} = \frac{\partial \bar{u}_i}{\partial t} \tag{2}
\]

Continuity at cylindrical surface of vertical drains:

\[
\frac{\partial^2 u_w}{\partial z^2} = -2 \frac{k_i}{r_w} \left. \left( \frac{\partial u_{i1}}{\partial r} \right) \right|_{r=r_w} \tag{3}
\]

Average pore pressure at the same depth:

\[
\bar{u}_i = \frac{1}{\pi (r_e^2 - r_w^2)} \left[ \int_{r_w}^{r_e} 2\pi ru_{i1}dr + \int_{r_e}^{r} 2\pi ru_{i1}dr \right] \tag{4}
\]

For the unimproved section \((h_i \leq z \leq H)\),

\[
\frac{k_v}{m_i \gamma_w} \frac{\partial^2 u_r}{\partial z^2} = \frac{\partial u_r}{\partial t} \tag{5}
\]
The continuity conditions at the plane of penetration \((z = h_1)\), are:

For the open-ended vertical drains:

\[
\begin{align*}
    u_w &= u_2 	ag{6a} \\
    -u_1 &= u_2 	ag{7a}
\end{align*}
\]

\[
\left(n^2 - 1\right) k_v \frac{\partial u_1}{\partial z} + k_w \frac{\partial u_w}{\partial z} = n^2 k_v \frac{\partial u_2}{\partial z} 	ag{8a}
\]

For the closed-ended vertical drains:

\[
\begin{align*}
    \frac{\partial u_w}{\partial z} &= 0 	ag{6b} \\
    -u_1 &= u_2 	ag{7b}
\end{align*}
\]

\[
\left(n^2 - 1\right) k_v \frac{\partial u_1}{\partial z} = n^2 k_v \frac{\partial u_2}{\partial z} 	ag{8b}
\]

The initial condition is:

\[
t = 0 \ , \quad -u_1(z) = u_0(z) = q_0 \quad \text{and} \quad u_2(z) = u_0(z) = q_0 \tag{9}
\]

where: \(u_{w1}(r,z,t)\) - pore pressure at any point in smear zone; \(u_{w1}(r,z,t)\) - pore pressure at any point in natural soil zone; \(\bar{u}_1(z,t)\) - average pore pressure of soil at the same depth in improved section; \(u_w(z)\) - pore pressure within vertical drains; \(u_z(z,t)\) - pore pressure of soil in unimproved section; \(h_1\) - length of vertical drains; \(r_w\) - radius of vertical drains; \(r_s\) - radius of smear zone; \(r_e\) - radius of influence zone of vertical drains; \(r\) - radial co-ordinate; \(z\) - vertical co-ordinate; \(t\) - time; \(m_v\) - coefficient of volume compressibility of the soil; \(k_v\) - vertical coefficient of permeability of soil; \(k_s\) - horizontal coefficient of permeability of remolded soil; \(\gamma_w\) - unit weight of water; \(u_0(z)\) - initial pore pressure; \(q_0\) - constant loading; \(n = r_e/r_w\).

3 SOLUTION

The solutions of \(u_w(z,t), \bar{u}_1(z,t)\) and \(u_z(z,t)\) can be expressed as follows (Tang and Onitsuka, 1998):

\[
u_w(z,t) = \sum_{m=0}^{\infty} Z_{wm1}(z) T_m(t) = \sum_{m=0}^{\infty} A_m g_{wm1}(z) e^{-\beta_m t} \tag{10}\]

\[
\bar{u}_1(z,t) = \sum_{m=0}^{\infty} Z_{m1}(z) T_m(t) = \sum_{m=0}^{\infty} A_m g_{m1}(z) e^{-\beta_m t} \tag{11}\]

\[
u_z(z,t) = \sum_{m=0}^{\infty} Z_{m2}(z) T_m(t) = \sum_{m=0}^{\infty} A_m g_{m2}(z) e^{-\beta_m t} \tag{12}\]
No matter whether the end of the vertical drains is open or closed, the following equation can be proved [3]:

\[
\int_{-1}^{1} y_m m v_z d z = \frac{(n^2 - 1) k_v Z'(h_1) Z_m(h_1) - k_w Z'(m)(h_1) Z_{m1}(h_1) + n^2 k_v Z'(m)(h_1) Z_{m2}(h_1) + \left[ (n^2 - 1) k_v Z'(m)(h_1) Z_{m1}(h_1) + k_w Z'(m)(h_1) Z_{m1}(h_1) - n^2 k_v Z'(m)(h_1) Z_{m2}(h_1) \right]}{n^2 - 1}
\]

For the case with open-ended vertical drains, substituting the Eqs. (10) ~ (12) into the continuity conditions at the plane of penetration Eqs. (6a) ~ (8a), the following three equations are obtained:

\[
Z_{m1}(h_1) = Z_{m2}(h_1) \tag{14a}
\]

\[
Z_{m1}(h_1) = Z_{m2}(h_1) \tag{15a}
\]

\[
(n^2 - 1)k_v Z'(m)(h_1) + k_w Z'(m)(h_1) = n^2 k_v Z'(m)(h_1) \tag{16a}
\]

Similarly, for the case with open-ended vertical drains, substituting the Eqs. (10) ~ (12) into the continuity conditions at the plane of penetration Eqs. (6b) ~ (8b), the following three equations are obtained:

\[
Z_{m1}(h_1) = 0 \tag{14b}
\]

\[
Z_{m1}(h_1) = Z_{m2}(h_1) \tag{15b}
\]

\[
(n^2 - 1)k_v Z'(m)(h_1) = n^2 k_v Z'(m)(h_1) \tag{16b}
\]

No matter whether the end of the vertical drains is open or closed, substituting Eqs. (14a) ~ (16a) or Eqs. (14b) ~ (16b) to Eq. (13), the following orthogonality is always founded:
\[(n^2 - 1) \int_0^h g_{m_1}(z) g_{m_1}(z) dz + n^2 \int_0^h g_{m_2}(z) g_{m_2}(z) dz = 0 \quad m \neq m' \quad (17)\]

By virtue of the above orthogonal relation and the initial condition Eq. (9), the value of \( A_m \) can be obtained as follows:

\[
A_m = \frac{(n^2 - 1)u_0 \int_0^h g_{m_1}(z) dz + n^2 u_0 \int_0^h g_{m_2}(z) dz}{(n^2 - 1) \int_0^h g_{m_1}(z) dz + n^2 \int_0^h g_{m_2}(z) dz} \quad (18)
\]

Substituting Eqs. (10) – (12) to the continuity conditions at the plane of the penetration, Eqs. (6a) ~ (8a) or Eqs. (6b) ~ (8b), the following matrix equation can be obtained:

\[
\mathbf{SX}^T = \mathbf{0} \quad (19)
\]

where:

\[
\mathbf{S} = \begin{bmatrix}
    s_{11} & s_{12} & s_{13} \\
    s_{21} & s_{22} & s_{23} \\
    s_{31} & s_{32} & s_{33}
\end{bmatrix} \quad (20)
\]

\[
\mathbf{X} = \begin{bmatrix}
    1 & c_{m_1} & b_{m_2}
\end{bmatrix} \quad (21)
\]

The elements of matrix \( \mathbf{S} \) are listed as follows:

For the open-ended vertical drains:

\[
s_{11} = \sin(\lambda_{m_1} \rho) ; \quad s_{12} = \sinh(\xi_{m_1} \rho) ; \quad s_{13} = -\cos[\lambda_{m_2}(1-\rho)] ; \quad s_{21} = h s_{11} ; \quad s_{22} = \lambda s_{12} ; \quad s_{23} = s_{13} ; \\
\]

\[
s_{31} = k_v + (n^2 - 1) h k_v \lambda_{m_1} \cos(\lambda_{m_1} \rho) / n^2 ; \quad s_{32} = k_v + (n^2 - 1) \lambda k_v \xi_{m_1} \cosh(\xi_{m_1} \rho) / n^2 ; \\
\]

\[
s_{33} = -k_v \lambda_{m_2} \sin[\lambda_{m_2}(1-\rho)] ;
\]

For the close-ended vertical drains:

\[
s_{11} = \lambda_{m_1} \cos(\lambda_{m_1} \rho) ; \quad s_{12} = \xi_{m_1} \cosh(\xi_{m_1} \rho) ; \quad s_{13} = 0 ; \quad s_{21} = h \sin(\lambda_{m_1} \rho) ; \quad s_{22} = \lambda \sinh(\xi_{m_1} \rho) ; \\
\]

\[
s_{23} = \cos[\lambda_{m_2}(1-\rho)] ; \quad s_{31} = s_{11} h (n^2 - 1) / n^2 ; \quad s_{32} = s_{13} \lambda (n^2 - 1) / n^2 ; \quad s_{33} = -\lambda_{m_2} \sin[\lambda_{m_2}(1-\rho)] ;
\]

\[
\rho = h_i / H , \quad \text{penetration fraction}.
\]

In order to get unequal zero solutions of vector \( \mathbf{X} \), \( \det \mathbf{S} \) must be equal to zero. \( \beta_m \) can be obtained from \( \det \mathbf{S} = 0 \). Substituting every value of \( \beta_m \) to Eq. (19), a series of \( c_{m_1} \) and \( b_{m_2} \) can be obtained.

The overall average degree of consolidation is expressed as:

\[
\bar{U} = 1 - \frac{1}{u_0 H} \left[ \int_0^h u_1(z) dz + \int_0^h u_2(z) dz \right] \quad (22)
\]
4 CALCULATIONS AND ANALYSIS

Figs. 2 and 3 show that, the consolidation speed for the open end is faster than that for the closed end. The difference of the consolidation speed under different drainage conditions of the end of vertical drains will become large with increasing of the permeability of the vertical drains. The consolidation speed is not significantly affected by the diameter, whether the end of the vertical drains is open or closed.

Fig. 4 shows that in the improved section, the consolidation speed near the end of vertical drains is affected by the drainage conditions of the end of the vertical drains. However, this effect applies for the whole of the unimproved section. The FEM results agree with the results for the closed end of vertical drains very well.

Figure 2. Consolidation affected by end conditions of vertical drains with different permeability

Figure 3. Consolidation affected by end conditions of vertical drains with different diameter

Figure 4. Pore pressure along with depth affected by end conditions of vertical drains
5 CONCLUSIONS

1) The basic equations and the solution conditions under two kinds of drainage conditions of the end of the vertical drains are the same except the continuity conditions at the plane of penetration. The forms of the final solutions are the same, except the elements of the matrix. The final expressions of the orthogonalities are the same.

2) The consolidation speed for the open end is faster than that for the closed end. The difference will become large with the increase in the permeability of the vertical drains.

3) For the improved section, consolidation speed, only the part near the plane of penetration is affected by the drainage conditions of the end of vertical drains. However, this effect applies for the whole of the unimproved section. The consolidation speed is not significantly affected by the diameter of the vertical drains.

REFERENCES